Charged particles in crossed and longitudinal electromagnetic fields and beam guides.

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Abstract

We consider a class of electromagnetic fields that contains crossed fields combined with longitudinal electric and magnetic fields. We study the motion of a classical particle (solutions of the Lorentz equations) in such fields. Then, we present an analysis that allows one to decide which fields from the class act as a beam guide for charged particles, and we find some time-independent and time-dependent configurations with beam guiding properties. We demonstrate that the Klein-Gordon and Dirac equations with all the fields from the class can be solved exactly. We study these solutions, which were not known before, and prove that they form complete and orthogonal sets of functions.

1 Introduction

Relativistic wave equations (Dirac and Klein–Gordon) provide a basis for relativistic quantum mechanics and QED of spinor and scalar particles. In relativistic quantum mechanics, solutions of relativistic wave equations are referred to as one-particle wave functions of fermions and bosons in external electromagnetic fields. In QED, such solutions permit the development of the perturbation expansion known as the Furry picture, which incorporates the interaction with the external field exactly, while treating the interaction with the quantized electromagnetic field perturbatively [1, 2, 3, 4, 5]. The most important exact solutions of the Klein–Gordon and Dirac equations are: solutions with the Coulomb field, which allow one to construct the relativistic theory of atomic spectra [6], solutions with a uniform magnetic field, which provide the basis of synchrotron radiation theory [7], and solutions in the field of a plane wave, which are widely used for calculations of quantum effects involving electrons and other elementary particles in laser beams [8]. Another physically important class of field configurations (for solving the relativistic wave equations) is a superposition of crossed fields and longitudinal fields. Solutions of relativistic equations with fields of this type were first studied by Redmond [9]. The Redmond configuration is a plane-wave combined with a constant longitudinal magnetic field. The corresponding solutions have wide spread applications, for example, in plasma physics [10] and cyclotron resonance [11]. In the works [12, 13, 14] exact solutions of the relativistic wave equations with a generalized Redmond configuration (Redmond field plus longitudinal electric fields) were found and used to calculate different quantum effects. In [15] the author has presented another generalization of a crossed field, which is a particular (the simplest) case of a vortex field [16, 17] (electromagnetic waves with vortices play a central role in singular optics [18]). He studied exact solutions of relativistic wave equations in such a field and he has discovered that it can be used to create a beam guide for charged particles.

In the present article we represent and study new solutions of the Klein–Gordon and Dirac equations with a new class of fields, which is a combination of crossed and longitudinal electromagnetic fields. For the crossed fields $E_z = H_z = 0$, $E_x = H_y$ and $E_y = -H_x$, and they depend on the time t and on the coordinate z via a light-cone variable $\xi = ct - z$. In the general case, the amplitudes of the crossed fields can also contain a linear ξ-dependent combination of the coordinates x, y. Thus, we can interpret the crossed fields as plane-waves with amplitudes linearly dependent on the coordinates x, y. One ought to say that this combination of crossed and longitudinal fields form a class which is described by several arbitrary

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ξ-dependent functions. This combination of electromagnetic fields is physically interesting, since some configurations act as beam guides for charged particles in a similar fashion to which the vortex field acts in the work [15]. The aforementioned vortex field is a particular case of our beam guiding configurations. It is interesting to stress that all other beam-guiding field configurations do not belong to the vortex field class.

The article is organized as follows: first, we describe potentials for the above mentioned combination of electromagnetic fields and we study classical particle motion (that is, solutions of the Lorentz equations) in such fields. Then, we present an analysis that allows one to decide which fields from the combination act as a beam guide for charged particles. We find some time-independent and time-dependent configurations with beam guiding properties. Finally, we study solutions of the Klein-Gordon and Dirac equations containing all the fields from the combination and we prove that these solutions form complete and orthogonal sets of functions. In the Appendix, we place some technical results.

The electromagnetic fields we consider are defined by the following potentials[1]:

$$
\begin{aligned}
&A^0 = \frac{1}{2}\left[\mathcal{G}(\xi) - A\right],\ A^1 = A_x = -\mathcal{F}_1(\xi) - \mathcal{H}(\xi)\, y\,, \\
&A^2 = A_y = -\mathcal{F}_2(\xi) + \mathcal{H}(\xi)\, x,\ A^3 = A_z = -\frac{1}{2}\left[\mathcal{G}(\xi) + A\right]\,,
\end{aligned} \tag{1}
$$

where

$$
A = R_{11}(\xi)\, x^2 + 2R_{12}(\xi)\, xy + R_{22}(\xi)\, y^2,\ \xi = x^0 - z = ct - z\,,
$$

and $\mathcal{G}(\xi),\ \mathcal{H}(\xi),\ \mathcal{F}_i(\xi),\ R_{i\,j}(\xi) = R_{j\,i}(\xi)\,,\ i,j = 1,2,$ are arbitrary functions of ξ. The corresponding electromagnetic fields have the form

$$
\begin{aligned}
&E_x = H_y = \mathcal{F}_1'(\xi) + R_{11}(\xi)\, x + \left[R_{12}(\xi) + \mathcal{H}'(\xi)\right] y,\ E_z = \mathcal{G}'(\xi)\ , \\
&E_y = -H_x = \mathcal{F}_2'(\xi) + \left[R_{12}(\xi) - \mathcal{H}'(\xi)\right] x + R_{22}(\xi)\, y,\ H_z = 2\mathcal{H}(\xi)\ .
\end{aligned} \tag{2}
$$

They consist of crossed fields and longitudinal electric and magnetic fields propagating along the z-axis. In the general case, amplitudes of the crossed fields depend linearly on the coordinates x, y.

The Maxwell current determined by the field (2) has the form

$$
j^\mu = \frac{c}{4\pi}(\rho, 0, 0, \rho)\,,\ \rho = \rho(\xi) = R_{11}(\xi) + R_{22}(\xi) - \mathcal{G}''(\xi)\ . \tag{3}
$$

2 Classical motion

Let us first examine the classical Lorentz equations

$$
m_0 c^2 \dot{u}^0 = e\left(\mathbf{uE}\right),\ m_0 c^2 \dot{\mathbf{u}} = e\mathbf{E}u^0 + e\left[\mathbf{u}\times\mathbf{H}\right],\ \left(u^0\right)^2 - \mathbf{u}^2 = 1\ , \tag{4}
$$

where

$$
u^\mu = \frac{dx^\mu}{ds} = \dot{x}^\mu = \left(u^0, \mathbf{u}\right),\ ds^2 = \eta_{\mu\,\nu} dx^\mu dx^\nu,\ \eta_{\mu\,\nu} = \mathrm{diag}\ (1,-1,-1,-1)\ .
$$

And the Hamilton–Jacobi equation is

$$
\left(\partial_0 \mathcal{S} + \frac{e}{c}A^0\right)^2 - \left(\nabla\mathcal{S} - \frac{e}{c}\mathbf{A}\right)^2 - m_0^2 c^2 = 0\ , \tag{5}
$$

where $\mathcal{S}$ is the classical action. From equations (4) obviously follow the equations for the kinetic momenta $P^\mu = m_0 c\, u^\mu = \left(P^0, \mathbf{P}\right)$:

$$
m_0 c^2 \dot{P}^0 = e\left(\mathbf{PE}\right),\ m_0 c^2 \dot{\mathbf{P}} = eP^0\mathbf{E} + e\left[\mathbf{P}\times\mathbf{H}\right],\ \left(P^0\right)^2 - \mathbf{P}^2 = m_0^2 c^2\ . \tag{6}
$$

[1] The four-dimensional coordinates of a particle are denoted as $x^\mu = \left(x^0 = ct,\ x^1 = x,\ x^2 = y, x^3 = z\right),\ \mu = 0,1,2,3,$ where c is the speed of light. Contravariant and covariant four-vectors are often represented in the form

$$
\begin{aligned}
&a^\mu = \left(a^0, a^i\right) = \left(a^0, \mathbf{a}\right),\ \mathbf{a} = \left(a^i\right)\ ,\ a^1 = a_x,\ a^2 = a_y,\ a^3 = a_z\,, \\
&a_\mu = \eta_{\mu\nu}a^\nu\ ,\ a^\mu = \eta^{\mu\nu}a_\nu\ .
\end{aligned}
$$

Three-vectors are indicated by boldface letters.

In particular, from (6), with allowance made for (2), we easily obtain

$$m_0c^2\,\dot{P}_z = e\,(\mathbf{PE}) + e\left(P^0 - P_z\right)E_z\ . \tag{7}$$

Let us introduce the generalized momenta p_μ according to the well-known relations

$$P_\mu = p_\mu - \frac{e}{c}A_\mu,\ p_\mu = -\partial_\mu \mathcal{S}\ . \tag{8}$$

One can easily prove that the quantity

$$\Lambda = p^0 - p_z \tag{9}$$

is an integral of motion. Indeed, (8) implies

$$\Lambda = p^0 - p_z = P^{\,0} - P_z + \frac{e}{c}\mathcal{G}\left(\xi\right)\ . \tag{10}$$

Hence, we obtain

$$\dot{\Lambda} = \dot{P}^{\,0} - \dot{P}_z + \frac{e}{c}\mathcal{G}'\left(\xi\right)\dot{\xi} = \dot{P}^{\,0} - \dot{P}_z + \frac{e}{c}E_z\dot{\xi}\ . \tag{11}$$

Taking into account the obvious relation

$$\dot{\xi} = \dot{x}^0 - \dot{z} = u^0 - u_z = \frac{P^{\,0} - P_z}{m_0 c} \tag{12}$$

and the equations (6) and (7), we find that (11) implies $\dot{\Lambda} = 0$, which completes the proof.

Let us introduce the notation

$$\Lambda = \hbar\lambda,\ g\left(\xi\right) = \frac{e}{c\hbar}\mathcal{G}\left(\xi\right),\ p\left(\xi\right) = \lambda - g\left(\xi\right),\ m = \frac{m_0 c}{\hbar}\ . \tag{13}$$

Then (10) can be rewritten as

$$P^{\,0} - P_z = \hbar p\left(\xi\right)\ , \tag{14}$$

and (12) implies

$$m\dot{\xi} = p\left(\xi\right) \Longrightarrow s = \int \frac{m d\xi}{p\left(\xi\right)}\ , \tag{15}$$

which relates the proper time and the parameter ξ.

In what follows, we denote

$$r_{ij}\left(\xi\right) = r_{ji}\left(\xi\right) = \frac{e}{c\hbar}R_{ij}\left(\xi\right),\ F_i\left(\xi\right) = \frac{e}{c\hbar}\mathcal{F}_i\left(\xi\right),\ (i,j=1,2),\ H\left(\xi\right) = \frac{e}{c\hbar}\mathcal{H}\left(\xi\right),\ S = \frac{1}{\hbar}\mathcal{S}. \tag{16}$$

Let us also introduce a 2×2 symmetric matrix $r = r\left(\xi\right)$ and the two-dimensional columns $F = F\left(\xi\right)$ and v,

$$r = \begin{pmatrix} r_{11}\left(\xi\right) & r_{12}\left(\xi\right) \\ r_{12}\left(\xi\right) & r_{22}\left(\xi\right) \end{pmatrix},\ F = \begin{pmatrix} F_1\left(\xi\right) \\ F_2\left(\xi\right) \end{pmatrix},\ v = \begin{pmatrix} x \\ y \end{pmatrix}\ . \tag{17}$$

The complete integral of the Hamilton–Jacobi equations (5) for the fields (2) can be presented as

$$S = -\frac{1}{2}\left[\lambda\left(x^0 + z\right) + \Gamma\right],\ \Gamma = v^+ f v + \chi^+ v + v^+\chi + F^+ v + v^+ F + \int\left(\chi^+\chi + m^2\right)p^{-1}\left(\xi\right)d\xi\ , \tag{18}$$

where the 2×2 real symmetric matrix $f = f\left(\xi\right)$,

$$f = \begin{pmatrix} f_{11}\left(\xi\right) & f_{12}\left(\xi\right) \\ f_{12}\left(\xi\right) & f_{22}\left(\xi\right) \end{pmatrix},$$

and the real two-column $\chi = \chi\left(\xi\right)$ satisfy the equations (see Appendix I)

$$p\left(\xi\right)\left[f'\left(\xi\right) + r\left(\xi\right)\right] - \left[f\left(\xi\right) + iH\left(\xi\right)\sigma_2\right]\left[f\left(\xi\right) - iH\left(\xi\right)\sigma_2\right] = 0\ , \tag{19}$$

$$p\left(\xi\right)\left[\chi'\left(\xi\right) + F'\left(\xi\right)\right] - \left[f\left(\xi\right) + iH\left(\xi\right)\sigma_2\right]\chi\left(\xi\right) = 0\ . \tag{20}$$

Here, σ_2 is a Pauli matrix.

Using (19) and (20), we can see that the three independent functions $f_{ij}(\xi)$ provide a solution to a set of three first-order non-linear equations, while the two functions $\chi_i(\xi)$ obey a set of two linear first-order inhomogenous equations, where the functions $f_{ij}(\xi)$ are assumed to be known. One should look for a particular solution of equations (19), and so the general solution of (20) has the structure

$$\chi(\xi) = k_1\chi^{(1)}(\xi) + k_2\chi^{(2)}(\xi) + \bar{\chi}(\xi) \ . \tag{21}$$

Here, $\bar{\chi}(\xi)$ is a particular solution for the set of inhomogenous equations (20); $\chi^{(1)}(\xi)$ and $\chi^{(2)}(\xi)$ provide a fundamental system of solutions for the set of homogeneous equations (20); k_1 and k_2 are arbitrary constants (two integrals of motion). Thus, the complete integral (18) of the Hamilton–Jacobi equations (5) depends on three integrals of motion, λ, k_1 and k_2.

Having at one's disposal solutions of the equations (19) and (20), one can easily find first integrals of the Lorentz equations. Using (4), with allowance made for (15), one readily obtains a set of equations for the coordinates x and y as functions of the variable ξ in the following matrix form:

$$p(\xi)\,v''(\xi) + p'(\xi)\,v'(\xi) - [r(\xi) + iH'(\xi)\,\sigma_2]\,v(\xi) - 2iH(\xi)\,v'(\xi) - F'(\xi) = 0 \ . \tag{22}$$

One can easily prove that this equation can be integrated once,

$$p(\xi)\,v'(\xi) + [f(\xi) - iH(\xi)\,\sigma_2]\,v(\xi) + \chi(\xi) = 0 \tag{23}$$

(see Appendix II).

Using identity (6) for the kinetic momenta, relations (14) and (16), we find

$$\begin{aligned}(P^0 - P_z)(P^0 + P_z) &= m_0^2c^2 + P_x^2 + P_y^2 \Longrightarrow (P^0 - P_z)(P^0 - P_z + 2P_z) \\ &= m_0^2c^2 + P_x^2 + P_y^2 \Longrightarrow p^2(\xi)\,[1 + 2z'(\xi)] = m^2 + p^2(\xi)\,v'^{+}(\xi)\,v'(\xi) \ ,\end{aligned}$$

the coordinate z being a function of the variable ξ. We finally obtain

$$2z'(\xi) - m^2p^{-2}(\xi) - v'^{+}(\xi)\,v'(\xi) + 1 = 0 \ . \tag{24}$$

Expressions (23) and (24) are first integrals of the Lorentz equations.

3 Crossed fields and beam guides

In this section we study a particular case of the field (2) in the absence of the longitudinal field, i.e., pure crossed-fields, and discuss how these kinds of fields can be used to create a beam guide for charged particles, i.e., fields that limit the motion of the charge around some given trajectories. These guides trap the charge in a bidimensional plane perpendicular to its trajectory and they are commonly used in many practical applications, e.g., quantum computation [19], high resolution spectroscopy [20], non-neutral plasma physics [21], and mass spectroscopy [22]. We will demonstrate that the Lorentz equations for these crossed-fields can be reduced to the classical Newton equation with a bidimensional effective potential. As an example we discuss a beam guide created by an electromagnetic vortex [15]. Different from the approximated classical analogues used to explain the operation of some RF traps [23], the analysis developed here is exact and can be used to describe the precise relativistic motion of the charge.

In the absence of the longitudinal field, we have

$$E_z = \mathcal{G}'(\xi) = H_z = \mathcal{H}(\xi) = 0 \ . \tag{25}$$

As the only influence of the constant $\mathcal{G}$ manifests itself through the z-component of the electric field, we can set $\mathcal{G} = 0$. So our potential (1) takes the form

$$\begin{aligned}&A^0 = A_z = -\frac{1}{2}A,\ A_x = -\mathcal{F}_1(\xi)\,,\ A_y = -\mathcal{F}_2(\xi) \ , \\ &A = R_{11}(\xi)\,x^2 + 2R_{12}(\xi)\,xy + R_{22}(\xi)\,y^2,\ \xi = x^0 - z \ .\end{aligned} \tag{26}$$

Whence, the fields

$$\begin{aligned}E_x = H_y &= \mathcal{F}_1'(\xi) + R_{11}(\xi)\,x + R_{12}(\xi)\,y \ , \\ E_y = -H_x &= \mathcal{F}_2'(\xi) + R_{12}(\xi)\,x + R_{22}(\xi)\,y \ .\end{aligned} \tag{27}$$

For these fields we can identify the integral of motion (10) with the light-front energy $E = \lambda/m$, and the parameter ξ is directly proportional to the proper time s, (15) $s = m\xi/\lambda$, with λ and m given by (13). Substituting the fields (27) in the Lorentz equation (4) we obtain:

$$m\dot{\mathbf{P}}_\perp = \lambda \mathbf{P}'_\perp = e\lambda \mathbf{E}_\perp = -e\lambda \left(\boldsymbol{\nabla}_\perp A_0 + \partial_0 \mathbf{A}_\perp\right) \tag{28}$$

where the symbol $\perp$ indicates the perpendicular x and y components of the vectors, e.g., $\boldsymbol{\nabla}_\perp = (\partial_x, \partial_y)$. We can eliminate the perpendicular components $\mathbf{A}_\perp$ of the potential (26) using the gauge transformation

$$\begin{aligned} &\tilde{A}_\mu(\xi, x, y) = A_\mu(\xi, x, y) + \partial_\mu \phi(\xi, x, y) \ , \\ &\phi(\xi, x, y) = x\mathcal{F}_1(\xi) + y\mathcal{F}_2(\xi) \ . \end{aligned}$$

So, making use of $\mathbf{P}'_\perp = \hbar\lambda\mathbf{x}''_\perp$, the equation (28) becomes

$$\lambda \mathbf{x}''_\perp = -\frac{e}{\hbar}\boldsymbol{\nabla}_\perp \tilde{A}_0 \ . \tag{29}$$

We can identify the above expression with Newton's non-relativistic equation for the two-dimensional motion of a particle with effective mass λ moving in the effective potential

$$U(\xi, x, y) = \frac{e}{\hbar}\tilde{A}_0 = \frac{e}{\hbar}\left[x\mathcal{F}'_1(\xi) + y\mathcal{F}'_2(\xi) - \frac{1}{2}A(\xi, x, y)\right] \ . \tag{30}$$

Therefore, we can find fields that trap a charge in some point of the x, y-plane without explicitly solving the Lorentz equations, just by looking for functions $A, \mathcal{F}_1$ and $\mathcal{F}_2$ for which the associated potential U is capable of limiting the classical motion of a particle of mass λ around this point.

For the special case of a *plane-wave*, where $\mathbf{E} = \mathbf{E}(\xi)$ and $\mathbf{H} = \mathbf{H}(\xi)$, we have $A = 0$, which generates the effective potential

$$U(\xi, x, y) = \frac{e}{\hbar}\left[x\mathcal{F}'_1(\xi) + y\mathcal{F}'_2(\xi)\right] \ , \tag{31}$$

and consequently creates a force $\hbar\mathbf{F}_\perp = e(\mathcal{F}'_1, \mathcal{F}'_2)$ that does not depend on the x, y coordinates. So although a plane-wave may limit the motion of a charge around some point, it is not possible to fix the position of this point only by manipulating the fields.

3.1 Time-independent fields

For a *time-independent potential*, the boundary trajectories can be found by looking for the minima of the surface $U(x, y)$. These points can be found using the standard procedure to determine the maxima and minima of a function of several variables [24]. A point (x_0, y_0) will be an extreme if the first derivatives $\partial U/\partial x$ and $\partial U/\partial y$ vanish at this point, and this extreme will be a minimum if the second derivative $\partial^2 U/\partial x^2$ and the discriminant $D(x, y)$ are positive at (x_0, y_0),

$$D(x_0, y_0) = \left(\frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial y^2}\right) - \left(\frac{\partial^2 U}{\partial x \partial y}\right)^2 > 0 \ , \ \left.\frac{\partial^2 U}{\partial x^2}\right|_{x_0, y_0} > 0 \ . \tag{32}$$

In the case of a time-independent potential, the expression (30) for U assumes the form

$$U(x, y) = \frac{e}{\hbar}\left[xC_1 + yC_2 - \frac{1}{2}\left(x^2 R_{11} + 2xyR_{12} + y^2 R_{22}\right)\right] \ , \tag{33}$$

where C_i and R_{ij} $(i, j = 1, 2)$ are constants. So the condition (32) implies

$$R_{11} < 0 \text{ and } R_{11}R_{22} > (R_{12})^2 \ , \tag{34}$$

and, consequently, $R_{22} < 0$ and $\det R \neq 0$. Therefore, the minimum for the potential (33), under the above restrictions (which is the unique extreme point of U and, consequently, a global minimum), is the point

$$x_0 = \frac{R_{22}C_1 - R_{12}C_2}{\det R} \ , y_0 = \frac{R_{11}C_2 - R_{12}C_1}{\det R} \ .$$

However, the fields (27) associated to the potential (33),

$$\begin{aligned} &E_x = H_y = C_1 - x|R_{11}| - y|R_{12}| \ , \\ &E_y = -H_x = C_2 - x|R_{12}| - y|R_{22}| \ , \end{aligned}$$

correspond to a problem of a constant charge density in the x, y plane and a constant current density in the z direction, which is nonrealistic.

3.2 Periodic time-dependent fields

For a *periodic time-dependent potential*, and the special case of *linear* (or quasi-linear[2]) systems (which includes the case of quadratics potentials (30)), where we have

$$\boldsymbol{\nabla}_{\perp}\tilde{A}_0 = M\left(t\right)\mathbf{x}_{\perp} \ ,$$

with $M\left(t\right)$ a periodic time-dependent 2×2 matrix, the stability of the potential (30) can be studied using the Lyapunov criteria [25]. To use these criteria, we first substitute Newton's second order equation (29) by a pair of first order equations making $\mathbf{u}_{\perp} = \mathbf{x}_{\perp}'$, that turns (29) equivalent to

$$\mathbf{V}_{\perp}' = \Xi\mathbf{V}_{\perp} \ , \ \mathbf{V}_{\perp} = \begin{pmatrix} \mathbf{x}_{\perp} \\ \mathbf{u}_{\perp} \end{pmatrix} \ , \ \Xi\left(t\right) = \begin{pmatrix} 0 & I \\ M\left(t\right) & 0 \end{pmatrix} \ . \tag{35}$$

The motion is called stable around the point $(0,0)$ if, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that for arbitrary initials values $\mathbf{V}_{\perp}\left(0\right)$ with moduli less than δ the coordinates $\mathbf{V}_{\perp}\left(t\right)$ remain of moduli less than ε for all the time $t \geq 0$, i.e., the motion is stable if

$$\forall\varepsilon > 0, \ \exists\, \delta > 0 \therefore \left|\mathbf{V}_{\perp}\left(0\right)\right| < \delta \Longrightarrow \left|\mathbf{V}_{\perp}\left(t\right)\right| < \varepsilon \ \ (t \geq 0) \ .$$

For periodic time-dependent $M\left(t\right)$ there always exists a transformation $\mathcal{R}$ that leads to a static problem $\tilde{\Xi} = \mathcal{R}\Xi\mathcal{R}^{-1} = \text{const.}$ having the same stability character as Ξ (see [25], Vol. II, p.119). So, after applying this transformation, the stability of the system can be analyzed by finding the roots λ_k of the characteristic polynomial

$$\det\left(\tilde{\Xi} - \lambda I\right) = 0 \ . \tag{36}$$

The system is stable if:

1. $\mathrm{Re}\left(\lambda_k\right) \leq 0$, for all λ_k;
2. The pure imaginary characteristic values $\mathrm{Re}\left(\lambda_k\right) = 0$ (if any such exist) are simple roots.

If at least one of the above conditions is violated the system will be unstable.

Let us use the above procedure to analyze the fields generated by the functions (60) of the next section, for a pure crossed field ($H = 0$). In this case, we have

$$\begin{aligned} &R_{11}\left(\xi\right) = C_1 + C_2\cos\omega\xi, \ \ R_{22}\left(\xi\right) = C_1 - C_2\cos\omega\xi \ , \\ &R_{12}\left(\xi\right) = C_2\sin\omega\xi \ , \ \ \mathcal{F}_1 = \mathcal{F}_2 = 0 \ , \ \ \omega, C_{1,2} = \text{const.} \end{aligned}$$

Substituting these values in (30) we obtain the effective potential

$$U\left(\xi,x,y\right) = -\frac{e}{2\hbar}\left[\left(x^2+y^2\right)C_1 + \left(x^2-y^2\right)C_2\cos\omega\xi + 2xyC_2\sin\omega\xi\right] \ . \tag{37}$$

Changing to a rotating frame, that is, making the transformation,

$$\tilde{\mathbf{x}}_{\perp} = R\mathbf{x}_{\perp} \ , \ \ R\left(\xi\right) = \begin{pmatrix} \cos\left(\omega\xi/2\right) & \sin\left(\omega\xi/2\right) \\ -\sin\left(\omega\xi/2\right) & \cos\left(\omega\xi/2\right) \end{pmatrix} , \tag{38}$$

the potential (37) becomes the following time-independent expression:

$$\tilde{U}\left(\tilde{x},\tilde{y}\right) = -\frac{e}{2\hbar}\left(\left(\tilde{x}^2+\tilde{y}^2\right)C_1 + \left(\tilde{x}^2-\tilde{y}^2\right)C_2\right) \ . \tag{39}$$

In the rotating frame the equation of motion (29) becomes

$$\begin{aligned} &\tilde{\mathbf{x}}_{\perp}'' = \left(RMR^{-1} - R\left(R^{-1}\right)''\right)\tilde{\mathbf{x}}_{\perp} - 2R\left(R^{-1}\right)'\tilde{\mathbf{x}}_{\perp}' \ , \\ &M\left(\xi\right) = \frac{e}{\lambda\hbar}\begin{pmatrix} C_1 + C_2\cos\left(\omega\xi\right) & C_2\sin\left(\omega\xi\right) \\ C_2\sin\left(\omega\xi\right) & C_1 - C_2\cos\left(\omega\xi\right) \end{pmatrix} , \end{aligned} \tag{40}$$

[2] That is, special systems for which the non-linear terms can be neglected. see Chapter XIV of [25].

and the expression (35) assumes the form

$$\tilde{\mathbf{V}}_\perp' = \tilde{\Xi}\tilde{\mathbf{V}}_\perp\,,\ \tilde{\mathbf{V}}_\perp = \begin{pmatrix} \tilde{\mathbf{x}}_\perp \\ \tilde{\mathbf{u}}_\perp \end{pmatrix},\ \tilde{\Xi} = \begin{pmatrix} 0 & \mathbf{I} \\ RMR^{-1} - R\left(R^{-1}\right)'' & -2R\left(R^{-1}\right)' \end{pmatrix},$$

where, from (38) and (40), we see that $\tilde{\Xi}$ is the constant matrix

$$\tilde{\Xi} = \begin{pmatrix} 0 & \mathbf{I} \\ c_1 + c_2\sigma_3 + \omega^2/4 & i\sigma_2\omega \end{pmatrix},\ c_i = \frac{eC_i}{\lambda\hbar}\,,$$

where σ_i are the Pauli matrices. The roots λ_k of the characteristic polynomial (36) are

$$\begin{aligned} \lambda_1 &= -\lambda_2 = \frac{1}{2}\sqrt{4c_1 - \omega^2 + 4\sqrt{c_2^2 - \omega^2 c_1}}\,, \\ \lambda_3 &= -\lambda_4 = \frac{1}{2}\sqrt{4c_1 - \omega^2 - 4\sqrt{c_2^2 - \omega^2 c_1}}\,. \end{aligned} \tag{41}$$

Since each eigenvalue has a negative partner, the two Lyapunov conditions will be satisfied only if $\mathrm{Re}\,(\lambda_k) = 0$.

The vortex field analyzed in [15] is a special case of (37) for $C_1 = 0$ and $C_2 = B_0\omega$. In this case, the constant potential (39) becomes

$$\tilde{U}\,(\tilde{x}, \tilde{y}) = -\frac{eB_0\omega}{2\hbar}\left(\tilde{x}^2 - \tilde{y}^2\right)\,. \tag{42}$$

This potential describes the surface of a saddle that rotates (by 38) in the x, y plane with angular velocity $\omega/2$ in time ξ (or angular velocity $\Omega/2 = \omega\lambda/2m$ in the proper time s). The classical motion of a particle in such a rotating-saddle potential is well known [23]. However, we were able to obtain some information about the trajectories without really solving the equations of motion.

The condition $\mathrm{Re}\,(\lambda_k) = 0$ for the eigenvalues (41) related to the potential (42) determines the expression

$$|\omega| \geq \frac{4e}{\lambda\hbar}\,|B_0|\,. \tag{43}$$

This inequality gives us a condition for which the potential (42) generates bounded trajectories. The above result concurs with the condition obtained in [15] by solving Lorentz's equation (4) or the one obtained in [23] by solving the Newton's equation (29).

4 Solutions of Klein-Gordon and Dirac equations

Solutions of the Klein–Gordon equation $\Phi_{\lambda,\mathbf{k}}\,(\xi, \eta, x, y)$ for the fields (2), labeled by the three integrals of motion λ (see 13)) and $\mathbf{k} = (k_1, k_2)$ (see (21)), read:

$$\begin{aligned} &\Phi_{\lambda,\mathbf{k}}\,(\xi, \eta, x\, y) = N_0 p^{-1/2}\,(\xi)\,\sqrt{\Delta\,(\xi)}\exp\,(iS)\,, \\ &\Delta\,(\xi) = \det B\,(\xi)\,,\ B\,(\xi) = \begin{pmatrix} \chi_1^{(1)}\,(\xi) & \chi_1^{(2)}\,(\xi) \\ \chi_2^{(1)}\,(\xi) & \chi_2^{(2)}\,(\xi) \end{pmatrix}, \\ &\eta = x^0 + z = ct + z\,, \end{aligned} \tag{44}$$

where N_0 is a normalization factor and $\chi_{s'}^{(s)}\,(\xi)$ $(s,\ s' = 1, 2)$ are the spinor components $\chi^{(s)}\,(\xi)$ introduced in (21),

$$\chi^{(s)} = \begin{pmatrix} \chi_1^{(s)}\,(\xi) \\ \chi_2^{(s)}\,(\xi) \end{pmatrix},\ s = 1, 2\,. \tag{45}$$

This fact can be directly verified by taking into account that the function $\Delta\,(\xi)$ obeys the equation

$$p\frac{d\Delta}{d\xi} = (\mathrm{tr} f\,)\,\Delta\,, \tag{46}$$

which is a consequence of the uniform set (20). Indeed, the spinors $\chi^{(s)}$ obey the following equation:

$$\left(\chi^{(s)}\right)' = p^{-1}\left[f + iH\sigma_2\right]\chi^{(s)} \Longleftrightarrow \left(\chi^{(s)+}\right)' = p^{-1}\chi^{(s)+}\left[f - iH\sigma_2\right]\,. \tag{47}$$

The linear independence of the spinors $\chi^{(s)}$ implies that the matrix B from (44) is nonsingular, i.e.,

$$\Delta = \det B = \chi_1^{(1)}(\xi)\,\chi_2^{(2)}(\xi) - \chi_2^{(1)}(\xi)\,\chi_1^{(2)}(\xi) \neq 0\ . \tag{48}$$

The function Δ can be easily calculated. One can see that for real spinors $\chi^{(s)}$ the following relations hold

$$\Delta = i\chi^{(1)+}\sigma_2\chi^{(2)} \Longrightarrow \Delta' = i\left(\chi^{(1)+}\right)'\sigma_2\chi^{(2)} + i\chi^{(1)+}\sigma_2\left(\chi^{(2)}\right)'\ . \tag{49}$$

Then, using (47), we find

$$\Delta' = p^{-1}\chi^{(1)+}\left[if\sigma_2 + H + i\sigma_2 f - H\right]\chi^{(2)} = p^{-1}\chi^{(1)+}\left[if\sigma_2 + i\sigma_2 f\right]\chi^{(2)}\ .$$

With the help of an evident identity

$$if\sigma_2 + i\sigma_2 f = (\mathrm{tr}f\,)\, i\sigma_2\,,$$

we finally find (46).

Solutions of the Dirac equation $\Psi_{\lambda,\,\mathbf{k}}(\xi,\eta,x,y)$ for the fields in question can be presented in a block form by using the two-dimensional Pauli matrices:

$$\Psi_{\lambda,\,\mathbf{k}}(\xi,\eta,x,y) = Np^{-1}(\xi)\sqrt{\Delta(\xi)}\exp\left[iS\right]\begin{pmatrix} m + p(\xi) - \sigma_3(\sigma\mathcal{F}) \\ \left[m - p(\xi)\right]\sigma_3 - (\sigma\mathcal{F}) \end{pmatrix} V(\xi)\ . \tag{50}$$

Here, the two-component spinor $V(\xi)$ reads

$$V(\xi) = \left[\cos T(\xi) + i\sigma_3 \sin T(\xi)\right] V_0\ , \tag{51}$$

where V_0 is an arbitrary constant two-component spinor, and we also denote

$$T(\xi) = \int H(\xi)\,p^{-1}(\xi)\ d\xi\ . \tag{52}$$

The components $\mathcal{F}_i$, $i = 1,2,3$ of the vector $\mathcal{F}$ have the form

$$\mathcal{F}_1 = f_{11}(\xi)\,x + \left[f_{12}(\xi) - H(\xi)\right]y + \chi_1(\xi)\,,\ \mathcal{F}_2 = \left[f_{12}(\xi) + H(\xi)\right]x + f_{22}(\xi)\,y + \chi_1(\xi)\,,\ \mathcal{F}_3 = 0\ .$$

Therefore, the classical and quantum-mechanical problems are reduced to the solution of the equations (19) and (20).

We will demonstrate that for a complete solution of the problem it is sufficient to find a special particular solution of equations (19).

The non-linear set of equations (19) can be linearized by the following substitution:

$$f(\xi) = p(\xi)\left[\cos T(\xi) + i\sigma_2 \sin T(\xi)\right] Z'(\xi)\,Z^{-1}(\xi)\left[\cos T(\xi) - i\sigma_2 \sin T(\xi)\right]\ , \tag{53}$$

where $Z(\xi)$ is a non-degenerate second-order matrix. Using (19), we find a linear second-order equation for the matrix $Z(\xi)$,

$$\begin{gathered} p^2(\xi)\,Z''(\xi) + p(\xi)\,p'(\xi)\,Z'(\xi) + \left[H^2(\xi) - p(\xi)\,\bar{r}(\xi)\right]Z(\xi) = 0\ , \\ \bar{r}(\xi) \equiv \left[\cos T(\xi) - i\sigma_2 \sin T(\xi)\right] r(\xi)\left[\cos T(\xi) + i\sigma_2 \sin T(\xi)\right]\ . \end{gathered} \tag{54}$$

A direct calculation yields

$$\begin{aligned} \bar{r}_{11}(\xi) &= \frac{1}{2}r_{11}(\xi)\left[1 + \cos 2T(\xi)\right] + \frac{1}{2}r_{22}(\xi)\left[1 - \cos 2T(\xi)\right] - r_{12}(\xi)\sin 2T(\xi)\ , \\ \bar{r}_{12}(\xi) &= \bar{r}_{21}(\xi) = r_{12}(\xi)\cos 2T(\xi) + \frac{1}{2}\left[r_{11}(\xi) - r_{22}(\xi)\right]\sin 2T(\xi)\ , \\ \bar{r}_{22}(\xi) &= \frac{1}{2}r_{22}(\xi)\left[1 + \cos 2T(\xi)\right] + \frac{1}{2}r_{11}(\xi)\left[1 - \cos 2T(\xi)\right] + r_{12}(\xi)\sin 2T(\xi)\ . \end{aligned} \tag{55}$$

In order that the matrix $f(\xi)$ be real and symmetric, one has to look for real solutions of the equations (54) that obey the subsidiary condition (the symbol $\sim$ stands for transposition)

$$J(\xi) = \tilde{J}(\xi)\,,\ J(\xi) \equiv Z'(\xi)\,Z^{-1}(\xi)\ . \tag{56}$$

Such solutions of (54) always exist, since, in accordance with (54), one easily finds an equation for $J(\xi)$,

$$p^2(\xi)\,J'(\xi) + p^2(\xi)\,J^2(\xi) + p(\xi)\,p'(\xi)\,J(\xi) + H^2(\xi) - p(\xi)\,\bar{r}(\xi) = 0\ . \tag{57}$$

Transposing this equation and using the property $\tilde{\bar{r}}(\xi) = \bar{r}(\xi)$, we find that the equations for $J(\xi)$ and $\tilde{J}(\xi)$ are the same, which proves the existence of solutions that satisfy (56).

Having at one's disposal a particular solution of equations (54) that satisfies the condition (56), one can integrate the equations (20) and (23) by quadratures. Indeed, a direct verification shows that the expressions

$$\begin{aligned} v &= [\cos T(\xi) + i\sigma_2 \sin T(\xi)]\, Z(\xi) \left\{ v_0 - \int Z^{-1}(\xi)\,[\cos T(\xi) - i\sigma_2 \sin T(\xi)]\,\chi(\xi)\,p^{-1}(\xi)\,d\xi \right\}\ , \\ \chi &= [\cos T(\xi) + i\sigma_2 \sin T(\xi)]\, \tilde{Z}^{-1}(\xi) \left\{ K - \int \tilde{Z}(\xi)\,[\cos T(\xi) - i\sigma_2 \sin T(\xi)]\,F'(\xi)\,d\xi \right\}\ , \end{aligned} \tag{58}$$

obey equations (20) and (23), respectively, where K is a column with components k_1 and k_2, and v_0 is a constant two-component spinor.

Motion in the fields of the type (2) has been studied in previous works. The authors of [26] found the symmetry operators for these fields, and the authors of [27, 28, 29, 30] found solutions for numerous specific fields of this kind. However, in all these specific fields there exist certain transformations that diagonalize $\bar{r}(\xi)$, so that the set of equations (54) splits into independent linear equations of second order.

A considerable progress was made in the work [15], which was the first to present exact solutions for a specific field that does not admit any transformations leading to the separation of equations (54) into independent equations of second order. The author of [15] examined a particular case of the fields (2) with the following choice of functions:

$$\mathcal{F}_i(\xi) = H(\xi) = g(\xi) = 0,\ r_{11}(\xi) = -r_{22}(\xi) = c\cos\omega\xi,\ r_{12}(\xi) = c\sin\omega\xi\ , \tag{59}$$

where ω and c are some constants. The solutions of the equations in [15] were obtained in a different manner from that of the approach of the present work, and they depend essentially on the specific form of the fields.

Let us note that in the present approach there is no necessity to assume $\mathcal{F}_i(\xi) = 0$, because these functions do not enter the set of equations (54), and so they may be left arbitrary.

We have succeeded in finding solutions for the fields defined by the following functions:

$$\begin{aligned} & g(\xi) = 0,\quad H(\xi) = H = \text{const},\quad r_{11}(\xi) = c_1 + c_2\cos\omega\xi\ , \\ & r_{22}(\xi) = c_1 - c_2\cos\omega\xi,\quad r_{12}(\xi) = c_2\sin\omega\xi,\quad \omega,\ c_{1,2} = \text{const}\ . \end{aligned} \tag{60}$$

A peculiarity of (60) is the presence of a constant and homogenous magnetic field. Expressions (59) provide a particular case of (60) with $H = c_1 = 0$.

It is easy to see that in the case of the functions (60), the equation (54) can be written in the form

$$\lambda^2 Z'' + \left[H^2 - \lambda c_1 - \lambda c_2(\sigma\mathbf{l})\right] Z = 0,\quad \mathbf{l} = (\sin\Omega\,\xi, 0, \cos\Omega\,\xi)\ , \tag{61}$$

where T is

$$T(\xi) = \frac{H\xi}{\lambda},\ \Omega = \omega + \frac{2H}{\lambda}\ .$$

For $c_2 = 0$, solutions of this equation are known [18, 28, 29, 30], and therefore we only need to examine the case $c_2 \neq 0$. A direct verification shows that the expressions

$$\begin{aligned} Z_{11} &= A\left[\alpha\cos\frac{\Omega\,x}{2}\cos\alpha\xi + \left(\frac{\Omega}{2} + \frac{\gamma + c_2}{\lambda\Omega}\right)\sin\frac{\Omega\,x}{2}\sin\alpha\xi\right]\ , \\ Z_{21} &= A\left[\alpha\sin\frac{\Omega\,x}{2}\cos\alpha\xi - \left(\frac{\Omega}{2} + \frac{\gamma + c_2}{\lambda\Omega}\right)\cos\frac{\Omega\,x}{2}\sin\alpha\xi\right]\ , \\ Z_{12} &= B\left[\beta\sin\frac{\Omega\,x}{2}\sin\beta\xi + \left(\frac{\Omega}{2} - \frac{\gamma + c_2}{\lambda\Omega}\right)\cos\frac{\Omega\,x}{2}\cos\beta\xi\right]\ , \\ Z_{22} &= -B\left[\beta\cos\frac{\Omega\,x}{2}\sin\beta\xi - \left(\frac{\Omega}{2} - \frac{\gamma + c_2}{\lambda\Omega}\right)\sin\frac{\Omega\,x}{2}\cos\beta\xi\right]\ , \end{aligned} \tag{62}$$

where

$$\alpha^2 = \frac{H^2 - \lambda c_1}{\lambda^2} + \frac{\Omega^2}{4} + \frac{\gamma}{\lambda},\ \beta^2 = \frac{H^2 - \lambda c_1}{\lambda^2} + \frac{\Omega^2}{4} - \frac{\gamma}{\lambda}\ ,$$
$$\gamma^2 = c_2^2 + (H^2 - \lambda c_1)\Omega^2,\quad A,\ B = \text{const.}$$

give a solution for the equations (61) that satisfies the condition (56). The signs of the quantities α, β, γ may be chosen arbitrarily. The quantity γ is either real or purely imaginary; the quantities α, β may be complex. For complex α, β, in view of the linear character of equations (61), the real and imaginary parts of (62) separately provide the sought solutions. The expressions (62) admit a continuous limiting process $\Omega \to 0$ in case the sign of γ (γ being real when $\Omega \to 0$) is chosen to obey the condition $c_2\gamma < 0$. By carrying out this limiting process (and redefining the constants A, B) we find, due to (62) and $\Omega = 0$, that

$$Z(\xi) = \begin{pmatrix} A\cos\alpha\xi & 0 \\ 0 & B\sin\beta\xi \end{pmatrix},\quad \begin{array}{l} \alpha^2 = \left(H^2 - \lambda c_1 - \lambda c_2\right)\lambda^{-2} \\ \beta^2 = \left(H^2 - \lambda c_1 + \lambda c_2\right)\lambda^{-2} \end{array}\ . \tag{63}$$

The calculation of the quantities $f(\xi)$, $\chi(\xi)$, $v(\xi)$, with the help of formulas (53) and (58), is reduced to simple algebraic manipulations and integrations of elementary functions, which we omit.

5 Orthogonality and completeness relations

In the case under consideration, it is convenient to define inner products for both scalar and spinor wave functions on the null-plane $\xi = \text{const}$, see for details [31, 32]. Such an inner product for scalar wave functions is

$$(\Phi_1,\ \Phi_2)_\xi = \int \left\{ \left[\hat{Q}\Phi_1(\xi,\eta,x,y)\right]^* \Phi_2(\xi,\eta,x,y) \right.$$
$$\left. + \Phi_1^*(\xi,\eta,x,y)\,\hat{Q}\Phi_2(\xi,\eta,x,y) \right\}\ d\eta dx dy\ , \tag{64}$$

where

$$\hat{Q} = \frac{\hat{P}_0 - \hat{P}_z}{\hbar} = 2i\frac{\partial}{\partial\eta} - g(\xi)\ .$$

For spinor wave functions, the inner product on the null-plane has the form

$$\langle\Psi_1,\ \Psi_2\rangle_\xi = \int \Psi_{1(-)}^{+}(\xi,\eta,x,y)\,\Psi_{2(-)}(\xi,\eta,x,y)\ d\eta dx dy\ ,$$
$$\Psi_{(-)} \hat{=} P_{(-)}\Psi,\ \hat{P}_{(-)} = \frac{1}{2}(1 - \alpha_3) = \frac{1}{2}\begin{pmatrix} I & -\sigma_3 \\ -\sigma_3 & I \end{pmatrix}\ . \tag{65}$$

One can verify that scalar wave functions (44) obey the orthonormality condition

$$(\Phi_{\lambda',\mathbf{k}'},\ \Phi_{\lambda,\mathbf{k}})_\xi = \varepsilon\,\delta(\lambda' - \lambda)\,\delta(k_1' - k_1)\,\delta(k_2' - k_2)\,,\ \varepsilon = p(\xi)\,|p(\xi)|^{-1}\ , \tag{66}$$

provided $N_0 = \left(32\pi^3\right)^{-1/2}$. First we note that the following relation holds

$$\hat{Q}\,\Phi_{\lambda,\mathbf{k}}(\xi,\eta,x,y) = p(\xi)\ \Phi_{\lambda,\mathbf{k}}(\xi,\eta,x,y)\ . \tag{67}$$

Then, integration over the variable η is reduced to the calculation of the integral

$$\int_{-\infty}^{\infty} \exp\left[\frac{i}{2}(\lambda' - \lambda)\eta\right]\ d\eta = 4\pi\delta(\lambda' - \lambda)\ . \tag{68}$$

Therefore, we can set $\lambda' = \lambda$ in the integral over x, y. The latter is reduced to the following integral

$$J = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ \exp\left[ix\sum_{s=1,2}\chi_1^{(s)}(\xi)(k_s' - k_s) + iy\sum_{s=1,2}\chi_2^{(s)}(\xi)(k_s' - k_s)\right]$$
$$= 4\pi^2\delta\left(\sum_{s=1,2}\chi_1^{(s)}(\xi)(k_s' - k_s)\right)\delta\left(\sum_{s=1,2}\chi_2^{(s)}(\xi)(k_s' - k_s)\right)\ . \tag{69}$$

The product of two δ-functions in the right hand side of (69) can be transformed if we take into account the following fact: Let a be a nonsingular 2×2 matrix, $\det a \neq 0$, with matrix elements a_{ij} . Then the relation holds

$$\delta\left(a_{11}z_1 + a_{12}z_2\right)\delta\left(a_{21}z_1 + a_{22}z_2\right) = |\det a|^{-1}\delta\left(z_1\right)\delta\left(z_2\right) \ . \tag{70}$$

Setting in (70): $a_{ij} = \chi_i^{(j)}\left(\xi\right),\ z_1 = k_1' - k_1,\ z_2 = k_2' - k_2$, we obtain

$$J = 4\pi^2 |\Delta\left(\xi\right)|^{-1}\delta\left(k_1' - k_1\right)\delta\left(k_2' - k_2\right) \ , \tag{71}$$

where $\Delta\left(\xi\right)$ is given by (48). Then the result (66) follows.

The constant spinor V_0 in solutions (51) is related to an additional (spinning) integral of motion, see for details [30, 31]. So the spinor V_0 (and therefore the Dirac wave function as well) depends on a spinning quantum number $\zeta = \pm 1$, $V_0 = V_0\left(\zeta\right)$. It is always possible to choose $V_0\left(\zeta\right)$ such that it obeys the following relations of orthonormality and completeness:

$$V_0^+\left(\zeta'\right)V_0\left(\zeta\right) = \delta_{\zeta,\zeta'}\,;\ \sum_{\zeta=\pm 1} V_0\left(\zeta\right)V_0^+\left(\zeta\right) = I \ . \tag{72}$$

Taking into account (72) and the relation

$$\Psi_{(-)\,\lambda,\mathbf{k},\zeta}\left(\xi,\eta,x,y\right) = \left(32\pi^3\right)^{-1/2}\Delta^{1/2}\left(\xi\right)\exp\left(iS\right)\begin{pmatrix} I \\ -\sigma_3 \end{pmatrix} V\left(\xi\right) \ , \tag{73}$$

we can verify that the spinor wave functions (51) obey the orthonormality condition

$$\langle \Psi_{\lambda',\mathbf{k}',\zeta'},\,\Psi_{\lambda,\mathbf{k},\zeta}\rangle_\xi = \delta\left(\lambda' - \lambda\right)\delta\left(k_1' - k_1\right)\delta\left(k_2' - k_2\right)\delta_{\zeta,\zeta'} \tag{74}$$

provided $N = \left(32\pi^3\right)^{-1/2}$.

The solutions (44) and (51) form complete sets of functions on the null-plane $\xi =$ const.

For scalar wave functions (44), we consider the following integral:

$$M = 2\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\ |p\left(\xi\right)|\,\Phi^*_{\lambda,\mathbf{k}}(\xi,\,\eta',\,x',\,y')\,\Phi_{\lambda,\mathbf{k}}\left(\xi,\eta,x,y\right) \ . \tag{75}$$

Integrating over the variables k_1, k_2 leads us to the integral:

$$\begin{aligned} M_1 &= \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\ \exp\left(iR^{(1)}k_1 + iR^{(2)}k_2\right) = 4\pi^2\delta\left(R^{(1)}\right)\delta\left(R^{(2)}\right) \ , \\ R^{(s)} &= \chi_1^{(s)}\left(\xi\right)\left(x' - x\right) + \chi_2^{(s)}\left(\xi\right)\left(y' - y\right),\ s = 1, 2 \ . \end{aligned}$$

This expression has the form (70), where $a = \tilde{B}\left(\xi\right)$ is the transpose of the matrix B in (44), and $z_1 = x' - x$, $z_2 = y' - y$. Thus we obtain:

$$M_1 = 4\pi^2 |\Delta\left(\xi\right)|^{-1}\delta\left(x' - x\right)\delta\left(y' - y\right) \ . \tag{76}$$

After that one can easily integrate over λ in (75) to get the following completeness relation:

$$\begin{aligned} &2\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\ |p\left(\xi\right)|\,\Phi^*_{\lambda,\mathbf{k}}\left(\xi,\eta',x',y'\right)\ \Phi_{\lambda,\mathbf{k}}\left(\xi,\eta,x,y\right) \\ &= \delta\left(x' - x\right)\delta\left(y' - y\right)\delta\left(\eta' - \eta\right) \ . \end{aligned}$$

Similar calculations can be performed in the spinor case. Here we have additionally to use the second relation (72) to get a completeness relation for the solutions (51):

$$\begin{aligned} &\sum_{\zeta=\pm 1}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2\ \Psi^+_{(-)\,\lambda,\mathbf{k},\zeta}\left(\xi,\eta',x',y'\right)\ \Psi_{(-)\,\lambda,\mathbf{k},\zeta}\left(\xi,\eta,x,y\right) \\ &= \delta\left(x' - x\right)\delta\left(y' - y\right)\delta\left(\eta' - \eta\right)\hat{P}_{(-)} \ . \end{aligned}$$

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Appendix

I. Equations (19) and (20) are obtained as follows. We search for a complete integral of the Hamilton–Jacobi equations (5) in the form

$$S=-\frac{1}{2}\left[\lambda\left(x^{0}+z\right)+\Gamma\right]\ , \tag{77}$$

with the function Γ is

$$\Gamma=f_{11}\left(\xi\right)x^{2}+2f_{12}\left(\xi\right)xy+f_{22}\left(\xi\right)y^{2}+2\left[\chi_{1}\left(\xi\right)+F_{1}\left(\xi\right)\right]x+2\left[\chi_{2}\left(\xi\right)+F_{2}\left(\xi\right)\right]y+\alpha\left(\xi\right)\ . \tag{78}$$

Here, $f_{ij}\left(\xi\right)$, $\chi_{i}\left(\xi\right)$, and $\alpha\left(\xi\right)$ are unknown functions of the variable ξ. Substituting the expression (77), with allowance made for (78), into equation (5), we obtain a quadratic form in x,y, with coefficients being functions of ξ, that must be identically zero, which is only possible when each coefficient is equal to zero. Hence, we obtain the following equations:

$$p\left(f_{11}^{\prime}+r_{11}\right)-f_{11}^{2}-\left(f_{12}+H\right)^{2}=0\ , \tag{79}$$
$$p\left(f_{22}^{\prime}+r_{22}\right)-f_{22}^{2}-\left(f_{12}-H\right)^{2}=0\ , \tag{80}$$
$$p\left(f_{12}^{\prime}+r_{12}\right)-f_{11}\left(f_{12}-H\right)-f_{22}\left(f_{12}+H\right)=0\ , \tag{81}$$
$$p\left(\chi_{1}^{\prime}+F^{\prime}_{1}\right)-f_{11}\chi_{1}-\left(f_{12}+H\right)\chi_{2}=0\ , \tag{82}$$
$$p\left(\chi_{2}^{\prime}+F^{\prime}_{2}\right)-f_{22}\chi_{2}-\left(f_{12}-H\right)\chi_{1}=0\ , \tag{83}$$
$$p\,\alpha^{\prime}-\chi_{1}^{2}-\chi_{2}^{2}-m^{2}=0\ . \tag{84}$$

Owing to (84), the expressions (18) and (78) for the function Γ are identical. It is easy to see that the set of equations (79)–(81) coincides with the matrix equation (19), and the set of equations (82)–(83) has the matrix form (20).

II. Let us show that the equation (22) is a consequence of the equation (23). To this end, we differentiate (23) with respect to the variable ξ,

$$pv^{\prime\prime}+p^{\prime}v^{\prime}+\left[f^{\prime}-iH^{\prime}\sigma_{2}\right]v+\left[f-iH\sigma_{2}\right]v^{\prime}+\chi^{\prime}=0\ . \tag{85}$$

However, the left-hand side of (85) satisfies the identity

$$\begin{aligned}&pv^{\prime\prime}+p^{\prime}v^{\prime}+\left[f^{\prime}-iH^{\prime}\sigma_{2}\right]v+\left[f-iH\sigma_{2}\right]v^{\prime}+\chi^{\prime}=A+B\ ,\\&A=pv^{\prime\prime}+p^{\prime}v^{\prime}-\left[r+iH^{\prime}\sigma_{2}\right]v-2iH\sigma_{2}v^{\prime}\ ,\\&B=\left[f^{\prime}+r\right]v+\left[f+iH\sigma_{2}\right]v^{\prime}+\chi^{\prime}\ .\end{aligned} \tag{86}$$

In the expression for B we now substitute $f^{\prime}$ from (19), $\chi^{\prime}$ from (20), and $v^{\prime}$ from (23), and thus we easily find that $B=-F^{\prime}$ and that $A+B$ is identical to the left-hand side of (22), which completes the proof.

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